4.1: The Preliminary Theory of Linear Equations

Highlights: We will use these analysis techniques for the rest of chapter 4

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Initial Conditions: $y(x_0) = y_0$

$y'(x_0) = y_1$

$\vdots$

$y^{(n-1)}(x_0) = y_{n-1}$

Theorem of existence of a unique solution

Let $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in the interval.

If $x = x_0$ is any point in the interval I, then a solution $y(x)$ of the initial value problem exists on the interval and is unique.

Interval of definition: (domain)

Homogenous Equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = \underline{0}$$

Non-homogenous Equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = \underline{g(x)}$$

$g(x) \neq 0$

Solutions of a homogeneous Equation

$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$

$c_1, c_2, \dots c_n$ are real coefficients

$y = 2x + 3x$

Definition of Linear Dependance and Independance

The set of functions $f_1(x), f_2(x), f_3(x), \dots, f_n(x)$ is linearly dependant if there exists constants $c_1, c_2, \dots, c_n$ (not all zero) such that

$$\underbrace{c_1 f_1(x) + c_2 f_2(x) + \dots}_{\text{Linear Combination}} c_n f_n(x) = 0$$

Ex) $x \quad -x \quad 1$

$c_1 x - c_2 x + c_3(1) = 0$

$c_1 = 1 \quad c_2 = 1 \quad c_3 = 0$

$x - x + 0 = 0$

$0 = 0$

$\square x^2 + \square e^x + \square \cos(x) = 0$

no combination of constants appears to work other than 0

$\Rightarrow$ Linearly Independant

$c_1 \cos^2 x + c_2 \sin^2 x + c_3(4) = 0$

$c_1 = 1 \quad c_2 = 1 \quad c_3 = -\frac{1}{4}$

$\Rightarrow$ Linearly Dependant

Linear Dependance occurs when you can write one function in terms of another.

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \dots + c_n f_n(x) = 0$$

If the linear combination above is zero only when $c_1 = c_2 = \dots = c_n = 0$, then the set of functions is <u>linearly</u> <u>independant</u>.

<u>Wronskians</u>

Suppose the functions $f_1(x), f_2(x), \dots f_n(x)$ have at least $n-1$ derivatives. Then the determinant

$$w(f_1, f_2, f_3, \dots, f_n)$$

$$= \begin{vmatrix} f_1 & f_2 & \dots\dots & f_n \\ f_1' & f_2' & \dots\dots & f_n' \\ f_1'' & f_2'' & \dots\dots & f_n'' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots\dots & f_n^{(n-1)} \end{vmatrix} = \det\left(\begin{bmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & \dots\dots & & f_n^{(n-1)} \end{bmatrix}\right)$$

Teorem: Criteria for linearly Independant Solutions

A set of functions $\{f_1, f_2, \dots f_n\}$ is linearly independant if and only if the wronskian $w(f_1, f_2, f_3, \dots, f_n) \neq 0$

$$\begin{array}{lll} f_1 = x & f_2 = -x & f_3 = 1 \\ f_1' = 1 & f_2' = -1 & f_3' = 0 \\ f_1'' = 0 & f_2'' = 0 & f_3'' = 0 \end{array}$$

$$w(x,-x,1) = \begin{vmatrix} x & -x & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$(2)(3) - (4)(-7) = 6 + 28 = 34$

3x3 shortcut

$$\begin{vmatrix} x & -x & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{matrix} x & -x \\ 1 & -1 \\ 0 & 0 \end{matrix}$$

$(0 + 0 + 0) - (0 + 0 + 0) = 0$

dependent since $w = 0$

$$\begin{vmatrix} x & -x & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} x & -x & 1 \\ & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} x & -x & 1 \\ & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} x & -x & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} x & x & 1 \\ 1 & - & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$x\begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} - -x\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= x(0-0) + x(0-0) + 1(0-0)$$

$$= 0$$